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Atoms in Boolean Powers

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We show that the Boolean power of an orthomodular lattice L is atomic if and only if both L and the underlying Boolean algebra B are atomic. We describe the set of all atoms in the Boolean power.

By an orthomodular lattice we will understand a bounded lattice L with the smallest element 0, the greatest element 1, and the orthocomplementation $\perp: L \to L$ fulfilling the following:

1. $a \leq b$ implies $b^\perp \leq a^\perp$ for each $a, b \in L$.
2. $(a^\perp)^\perp = a$ for each $a \in L$.
3. $a \vee a^\perp = 1$ for each $a \in L$.
4. If $a \leq b$, then $b = a \vee (b \wedge a^\perp)$ for each $a, b \in L$.

We will deal with Boolean powers of orthomodular lattices. There are several ways to define a Boolean power (see, e.g., Burris, 1975). The definition we use here was given in Drossos (1992), where some of its other properties were also shown. Other aspects of Boolean powers have been studied, e.g., in Banaschewski and Nelson (1979) and Riečanová (1992).

Definition 1. Let B be a complete Boolean algebra, and let L be an orthomodular lattice. The set

$$\mathbb{L}[\mathbb{B}] = \left\{ f: L \to B; f(t_1) \wedge f(t_2) = 0 \text{ if } t_1, t_2 \in L, t_1 \neq t_2; \bigvee_{t \in L} f(t) = 1 \right\}$$

is called a *Boolean power* of L (with respect to B).

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The partial order in $\mathbb{L}[\mathbb{B}]$ is defined as follows:

Definition 2. Let $f, g \in \mathbb{L}[\mathbb{B}]$. Then $f \leq g$ if and only if

$$\bigvee_{\substack{t_1, t_2 \in L \\ t_1 \leq t_2}} (f(t_1) \wedge g(t_2)) = 1$$

An equivalent expression for the partial order in $\mathbb{L}[\mathbb{B}]$ is given in the following lemma:

Lemma 3. Let $f, g \in \mathbb{L}[\mathbb{B}]$. Then $f \leq g$ if and only if for each $t_1, t_2 \in L$ the following holds: if $f(t_1) \wedge g(t_2) = 0$, then $t_1 \leq t_2$.

In the next definition we introduce an orthocomplementation into $\mathbb{L}[\mathbb{B}]$.

Definition 4. Let $f, g \in \mathbb{L}[\mathbb{B}]$. The function f is an *orthocomplement* of the function g ($f = g^{\perp}$) if for each $t \in L$ there is $f(t) = g(t^{\perp})$.

A technical verification shows that $\mathbb{L}[\mathbb{B}]$ with the partial order and the orthocomplementation defined as above is an orthomodular lattice. The zero element in $\mathbb{L}[\mathbb{B}]$ is the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0 \end{cases}$$

and the unit element in $\mathbb{L}[\mathbb{B}]$ is the function

$$g(x) = \begin{cases} 1 & \text{for} \quad x = 1 \\ 0 & \text{for} \quad x \neq 1 \end{cases}$$

An important class among the orthomodular lattices are the atomic ones. Before dealing with the question of atomicity for Boolean powers, we review the usual definition of an atom and an atomic lattice.

Definition 5. Let L be a lattice. A nonzero element $a \in L$ is called an *atom* in L if there is no $b \in L$ such that $b \leq a$, $b \neq 0$, $b \neq a$. If for each nonzero element $t \in L$ there is an atom $a \in L$ such that $a \leq t$, then L is said to be *atomic*.

The main result of this work is included in the following theorem.

Theorem 6. $\mathbb{L}[\mathbb{B}]$ is atomic if and only if both B and L are atomic. Each atom in $\mathbb{L}[\mathbb{B}]$ has the form

$$f(x) = \begin{cases} a & \text{for} \quad x = b \\ a^{\perp} & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0, \quad x \neq b \end{cases}$$

where a is an atom in B and b is an atom in L. Moreover, each element of this form is an atom in $\mathbb{L}[\mathbb{B}]$.

Proof. Let $\mathbb{L}[\mathbb{B}]$ be atomic and let a be a nonzero element in B; let b be a nonzero element in L. We will consider the following element of $\mathbb{L}[\mathbb{B}]$:

$$f(x)=\begin{cases} a & \text{for} \quad x=b \\ a^{\perp} & \text{for} \quad x=0 \\ 0 & \text{for} \quad x\neq 0, \quad x\neq b \end{cases}$$

As $\mathbb{L}[\mathbb{B}]$ is atomic, there exists an atom $f_0 \in \mathbb{L}[\mathbb{B}]$ such that $f_0 \leq f$. Using Lemma 3, we obtain that there are $b_0 \in L$, $a_0 \in B$ such that

$$f_0(x)=\begin{cases} a_0 & \text{for} \quad x=b_0 \\ a_0^{\perp} & \text{for} \quad x=0 \\ 0 & \text{for} \quad x\neq 0, \quad x\neq b_0 \end{cases}$$

A simple verification leads to the inequalities $a_0 \leq$, $b_0 \leq b$ and to the fact that a_0 and b_0 are atoms in B and L, respectively. Thus b and L are atomic.

Now let B and L be atomic and let f be a nonzero element of $\mathbb{L}[\mathbb{B}]$. Then there exist $a \in B$, $b \in L$, both nonzero, such that $f(b)=a$. Let a_0 be an atom in B, $a_0 \leq a$, let b_0 be an atom in L, $b_0 \leq b$. It is easy to see (Lemma 3) that for the function

$$f_0(x)=\begin{cases} a_0 & \text{for} \quad x=b_0 \\ a_0^{\perp} & \text{for} \quad x=0 \\ 0 & \text{for} \quad x\neq 0, \quad x\neq b_0 \end{cases}$$

we obtain that f_0 is an atom in $\mathbb{L}[\mathbb{B}]$ for which $f_0 \leq f$. Hence $\mathbb{L}[\mathbb{B}]$ is atomic and all its atoms are of the given form. Moreover, it is obvious that each such element is an atom in $\mathbb{L}[\mathbb{B}]$.

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